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Evolution Inclusions
and Variation Inequalities
for Earth Data Processing II

Differential-Operator Inclusions
and Evolution Variation Inequalities
for Earth Data Processing
Preface

By this book we continue the series of monographs devoted to investigation method for mathematical models of non-linear geophysical processes and fields. In the first volume the content of the second volume is announced.

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Dr. Mikhail Z. Zgurovsky
Valery S. Mel’nik
Pavlo O. Kasyanov
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# Acronyms

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
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<tbody>
<tr>
<td>a.e.</td>
<td>Almost everywhere</td>
</tr>
<tr>
<td>FG method</td>
<td>Faedo–Galerkin method</td>
</tr>
<tr>
<td>for a.e.</td>
<td>For almost each</td>
</tr>
<tr>
<td>l.s.c.</td>
<td>Lower semicontinuous</td>
</tr>
<tr>
<td>LTS</td>
<td>Linear topological space</td>
</tr>
<tr>
<td>$N$-s.b.v.</td>
<td>$N$-semibounded variation</td>
</tr>
<tr>
<td>$N$-sub-b.v.</td>
<td>$N$-subbounded variation</td>
</tr>
<tr>
<td>r.c.</td>
<td>Radial continuous</td>
</tr>
<tr>
<td>r.l.s.c.</td>
<td>Radial lower semicontinuous</td>
</tr>
<tr>
<td>r.s.c.</td>
<td>Radial semicontinuous</td>
</tr>
<tr>
<td>r.u.s.c.</td>
<td>Radial upper semicontinuous</td>
</tr>
<tr>
<td>s.b.v.</td>
<td>Semibounded variation</td>
</tr>
<tr>
<td>s.m.</td>
<td>Semimonotone</td>
</tr>
<tr>
<td>sub-m.</td>
<td>Submonotone</td>
</tr>
<tr>
<td>sub-b.v.</td>
<td>Subbounded variation</td>
</tr>
<tr>
<td>u.h.c.</td>
<td>Upper hemicontinuous</td>
</tr>
<tr>
<td>u.s.c.</td>
<td>Upper semicontinuous</td>
</tr>
<tr>
<td>u.s.b.v.</td>
<td>Uniform semibounded variation</td>
</tr>
<tr>
<td>$V$-s.b.v.</td>
<td>$V$-semibounded variation</td>
</tr>
<tr>
<td>w.l.s.c.</td>
<td>Weakly lower semicontinuous</td>
</tr>
</tbody>
</table>
Introduction

At an analysis and control of different geophysical and socio-economical processes it is often appears such problem: at a mathematical modelling of effects related to friction and viscosity, quantum effects, a description of different nature waves the existing “gap” between rather high degree of the mathematical theory of analysis and control for non-linear processes and fields and practice of its using in applied scientific investigations make us require rather stringent conditions for interaction functions. These conditions related to linearity, monotony, smoothness, continuity and can substantially have an influence on the adequacy of mathematical model.

Let us consider for example some diffusion process. Its mathematical model has the next form:

\[
\begin{align*}
    y_t - \Delta y + f(y) &= g(t, x) & \text{in} & & \Omega \times (\tau; T), \\
    y|_{\partial \Omega} &= 0, \\
    y|_{t=T} &= y_0,
\end{align*}
\]

(1)

here \( n \geq 2, \Omega \subset \mathbb{R}^n \) is a bounded domain with a rather smooth boundary, \(-\infty < \tau < T < +\infty, g : \Omega \times (\tau; T) \rightarrow \mathbb{R}, y_0 : \Omega \rightarrow \mathbb{R} \) are rather regular functions, \( f : \mathbb{R} \rightarrow \mathbb{R} \) is an interaction function, \( y : \Omega \times (\tau; T) \rightarrow \mathbb{R} \) is an unknown function.

It is well known that if \( f \) is a rather smooth function and satisfies for example the next condition of no more than polynomial growth:

\[
\exists p > 1, \quad \exists c > 0 : \quad |f(s)| \leq c(1 + |s|^{p-1}) \quad \forall s \in \mathbb{R},
\]

(2)

then problem (1) has a unique rather regular solution. Let us consider the case when \( f \) is continuous and initial data and external forces are nonregular (for example \( y_0 \in L_2(\Omega), g \in L_2(\Omega \times (\tau; T)) \)). Then, as a rule, we consider the generalized setting of problem (1):

\[
\begin{align*}
    y'(t) + A(y(t)) + B(y(t)) &= g(t) & \text{for a.e.} & & t \in (\tau; T), \\
    y(\tau) &= y_0,
\end{align*}
\]

(3)

here \( A : V_1 \rightarrow V_1^* \) is an energetic extension of operator \(-\Delta\), \( B : V_2 \rightarrow V_2^* \) is the Nemytskii operator for \( F, V_1 = H_0^1(\Omega) \) is a real Sobolev space, \( V_2 = L_p(\Omega), V_1^* = H^{-1}(\Omega), V_2^* = L_{q}(\Omega) \), \( q \) is the conjugated index, \( y' \) is a derivative of an
element \( y \in L_2(\tau, T; V_1) \cap L_p(\tau, T; V_2) \) and it is considered in the sense of the space \( \mathcal{D}^*_q([\tau; T], V_1^* + V_2^*) \).

A solution of problem (3) in the class \( W = \{ y \in L_2(\tau, T; V_1) \cap L_p(\tau, T; V_2) \mid y' \in L_2(\tau, T; V_1^*) + L_q(\tau, T; V_2^*) \} \) refers to be the generalized solution of problem (1).

To prove the existence of solutions for problem (1) as a rule we need to add supplementary “signed condition” for an interaction function \( f \), for example,

\[
\exists \alpha, \beta > 0 : \quad f(s)s \geq \alpha |s|^p - \beta \quad \forall s \in \mathbb{R}.
\]  

But we do not succeed in proving the uniqueness of the solution of such problem in the general case. Note that technical condition (4) provides a dissipation too. We remark also that different conditions for parameters of problem (1) provide corresponding conditions for generated mappings \( A \) and \( B \).

Problem (3) is usually investigated in more general case:

\[
\begin{cases}
y' + \mathcal{A}(y) = g, \\
y(\tau) = y_0,
\end{cases}
\]  

here \( \mathcal{A} : X \rightarrow X^* \) is the Nemytskii operator for \( A + B \),

\[
\mathcal{A}(y)(t) = A(y(t)) + B(y(t)) \quad \text{for a.e.} \quad t \in (\tau; T), \ y \in X,
\]

\[
X = L_2(\tau, T; V_1) \cap L_p(\tau, T; V_2), \quad X^* = L_2(\tau, T; V_1^*) + L_q(\tau, T; V_2^*).
\]

Solutions of problem (5) are also searched in the class \( W = \{ y \in X \mid y' \in X^* \} \).

In cases when the continuity of the interaction function \( f \) have an influence on the adequacy of mathematical model fundamentally then problem (1) is reduced to such problem:

\[
\begin{cases}
y_t - \Delta y + F(y) \ni g(x, t) \quad \text{in} \ Q = \Omega \times (\tau; T), \\
y|_{\partial \Omega} = 0, \\
y|_{\tau = T} = y_0,
\end{cases}
\]  

here

\[
F(s) = [\underline{f}(s), \overline{f}(s)], \quad \underline{f}(s) = \lim_{t \to s} f(t), \quad \overline{f}(s) = \lim_{t \to s} f(t), \quad s \in \mathbb{R},
\]

\[
[a, b] = \{ \alpha a + (1 - \alpha)b \mid \alpha \in [0, 1] \},
\]

\(-\infty < a < b < +\infty \).

A solution of such differential-operator inclusion

\[
\begin{cases}
y' + \mathcal{A}(y) \ni g, \\
y(\tau) = y_0,
\end{cases}
\]  

is usually thought to be the generalized solution of problem (6). Here \( \mathcal{A} : X \rightarrow X^* \),

\[ \mathcal{A}(u) = \{ p \in X^* \mid p(t) \in A(u(t)) + B(u(t)) \} \text{ for a.e. } t \in (\tau, T) \}, \quad u \in X, \]

\( A : V_1 \rightarrow V_1^* \) is the energetic extension of “\(-\Delta\)” in \( H_0^1(\Omega) \), \( B : V_2 \rightarrow C_c(V_2^*) \) is the Nemyskii operator for \( F \):

\[ B(v) = \{ z \in V_2^* \mid z(x) \in F(v(x)) \} \text{ for a.e. } x \in \Omega \}, \quad v \in V_2. \]

Taking into account all variety of classes of mathematical models for different nature geophysical processes and fields we propose rather general approach to investigation of them in this book. Further we will study classes of mathematical models in terms of general properties of generated mappings like \( \mathcal{A} \).

This monograph is the continuation of [ZMK10]. Let us consider some denotations and results, that we will use in this book. Let \( X \) be a Banach space, \( X^* \) be its topologically adjoint,

\[ \langle \cdot, \cdot \rangle_X : X^* \times X \rightarrow \mathbb{R} \]

be the canonical duality between \( X \) and \( X^* \), \( 2^{X^*} \) be a family of all subsets of the space \( X^* \), let \( A : X \rightarrow 2^{X^*} \) be the multivalued map,

\[ \text{graph} A = \{ (\xi; y) \in X^* \times X \mid \xi \in A(y) \}, \]

\[ \text{Dom} A = \{ y \in X \mid A(y) \neq \emptyset \}. \]

The multivalued map \( A \) is called strict if \( \text{Dom} A = X \). Together with every multivalued map \( A \) we consider its upper

\[ [A(y), \xi]_+ = \sup_{d \in A(y)} \langle d, \xi \rangle_X \]

and lower

\[ [A(y), \xi]_- = \inf_{d \in A(y)} \langle d, \xi \rangle_X \]

support functions, where \( y, \xi \in X \). Let also

\[ \| A(y) \|_+ = \sup_{d \in A(y)} \| d \|_{X^*}, \quad \| A(y) \|_- = \inf_{d \in A(y)} \| d \|_{X^*}, \quad \| \emptyset \|_+ = \| \emptyset \|_- = 0. \]

For arbitrary sets \( C, D \in 2^{X^*} \) we set

\[ \text{dist}(C, D) = \sup_{e \in C} \inf_{d \in D} \| e - d \|_{X^*}, \quad d_H(C, D) = \max \{ \text{dist}(C, D), \text{dist}(D, C) \}. \]

Then, obviously,

\[ \| A(y) \|_+ = d_H(A(y), 0) = \text{dist}(A(y), 0), \quad \| A(y) \|_- = \text{dist}(0, A(y)). \]
Together with the operator \( A : X \to 2^{X^*} \) let us consider the following maps

\[
\text{co} A : X \to 2^{X^*} \quad \text{and} \quad \text{co}^* A : X \to 2^{X^*},
\]
defined by relations

\[
(\text{co} A)(y) = \text{co}(A(y)) \quad \text{and} \quad (\text{co}^* A)(y) = \text{co}^*(A(y))
\]
respectively, where \( \text{co}^*(A(y)) \) is the weak star closure of the convex hull \( \text{co}(A(y)) \) for the set \( A(y) \) in the space \( X^* \). Besides for every \( G \subset X \)

\[
(\text{co} A)(G) = \bigcup_{y \in G}(\text{co} A)(y), \quad (\text{co}^* A)(G) = \bigcup_{y \in G}(\text{co}^* A)(y).
\]

Further we will denote the strong, weak and weak star convergence by \( \to, \rightharpoonup, \star \) or \( \to, \rightharpoonup, \rightharpoonup \) respectively. As \( C_b(X^*) \) we consider the family of all nonempty convex closed bounded subsets from \( X^* \).

**Proposition 1.** [ZMK10, Proposition 1.2.1] Let \( A, B, C : X \rightharpoonup X^* \). Then for all \( y, v, v_1, v_2 \in X \) the following statements take place:

1. The functional \( X \ni u \to [A(y), u]_+ \) is convex, positively homogeneous and lower semicontinuous;
2. \( [A(y), v_1 + v_2]_+ \leq [A(y), v_1]_+ + [A(y), v_2]_+ \),
   \( [A(y), v_1 + v_2]_- \geq [A(y), v_1]_- + [A(y), v_2]_- \),
   \( [A(y), v_1 + v_2]_+ \geq [A(y), v_1]_+ + [A(y), v_2]_- \),
   \( [A(y), v_1 + v_2]_- \leq [A(y), v_1]_+ + [A(y), v_2]_- \);
3. \( [A(y) + B(y), v]_+ = [A(y), v]_+ + [B(y), v]_+ \),
   \( [A(y) + B(y), v]_- = [A(y), v]_- + [B(y), v]_- \);
4. \( [A(y), v]_+ \leq \|A(y)\|_X + \|v\|_X \),
   \( [A(y), v]_- \leq \|A(y)\|_X - \|v\|_X \);
5. \( \|\text{co}^* A(y)\|_+ = \|A(y)\|_+ \), \( \|\text{co}^* A(y)\|_- = \|A(y)\|_- \),
   \[
   [A(y), v]_+ = \left[\text{co}^* A(y), v\right]_+, \quad [A(y), v]_- = \left[\text{co}^* A(y), v\right]_-;
   \]
6. \( \|A(y) - B(y)\|_+ \geq \|A(y)\|_+ - \|B(y)\|_- \),
   \( \|A(y) - B(y)\|_- \geq \|A(y)\|_- - \|B(y)\|_+ \);
7. \( d \in \text{co}^* A(y) \iff \forall \omega \in X \ [A(y), \omega]_+ \geq \langle d, \omega \rangle_X \);
8. \( d_H(A(y), B(y)) \geq \|A(y)\|_+ - \|B(y)\|_+ \),
   \( d_H(A(y), B(y)) \geq \|A(y)\|_- - \|B(y)\|_- \),
   where \( d_H \) is Hausdorff metric;
9. \( \text{dist}(A(y) + B(y), C(y)) \leq \text{dist}(A(y), C(y)) + \text{dist}(B(y), 0) \),
   \( \text{dist}(C(y), A(y) + B(y)) \leq \text{dist}(C(y), A(y)) + \text{dist}(0, B(y)) \),
   \( d_H(A(y) + B(y), C(y)) \leq d_H(A(y), C(y)) + d_H(B(y), 0) \);
10. for any $D \subset X^*$ and bounded $E \in C_0(X^*)$

$$\text{dist}(D, E) = \text{dist}^{*}(\overline{D}, E).$$

**Proposition 2.** [ZMK10, Proposition 1.2.2] The inclusion $d \in C_0^* A(y)$ holds true if and only if one of the following relations takes place:

- either $[A(y), v]_+ \geq (d, v)_X \quad \forall v \in X,$
- or $[A(y), v]_- \leq (d, v)_X \quad \forall v \in X.$

**Proposition 3.** [ZMK10, Proposition 1.2.3] Let $D \subset X$ and $a(\cdot, \cdot) : D \times X \to \mathbb{R}.$ For each $y \in D$ the functional $X \ni w \mapsto a(y, w)$ is positively homogeneous, convex and lower semicontinuous if and only if there exists the multivalued map $A : X \to 2^{X^*}$ such that $D(A) = D$ and

$$a(y, w) = [A(y), w]_+ \quad \forall y \in D(A), w \in X.$$

**Proposition 4.** [ZMK10, Proposition 1.2.4] The functional $\| \cdot \|_+ : C_0(X^*) \to \mathbb{R}_+$ satisfies the following properties:

1. $\{ \emptyset \} = A \quad \iff \quad \| A \|_+ = 0,$
2. $\| \alpha A \|_+ = |\alpha| \| A \|_+, \quad \forall \alpha \in \mathbb{R}, A \in C_0(X^*),$
3. $\| A + B \|_+ \leq \| A \|_+ + \| B \|_+ \quad \forall A, B \in C_0(X^*).$

**Proposition 5.** [ZMK10, Proposition 1.2.5] The functional $\| \cdot \|_- : C_0(X^*) \to \mathbb{R}_+$ satisfies the following properties:

1. $\{ \emptyset \} = A \quad \iff \quad \| A \|_- = 0,$
2. $\| \alpha A \|_- = |\alpha| \| A \|_- , \quad \forall \alpha \in \mathbb{R}, A \in C_0(X^*),$
3. $\| A + B \|_- \leq \| A \|_- + \| B \|_- \quad \forall A, B \in C_0(X^*).$

Let us consider some classes of multivalued maps that we introduced in [ZMK10]. As before, let $X$ be a Banach space, $X^*$ be its topologically adjoint,

$$\langle \cdot, \cdot \rangle_X : X^* \times X \to \mathbb{R}$$

be the duality form (Fig. 1).

We remind that the multivalued map $A : D(A) \subset X \to 2^{X^*}$ is called the monotone one if

$$\langle d_1 - d_2, y_1 - y_2 \rangle_X \geq 0 \quad \forall y_1, y_2 \in D(A), \forall d_1 \in A(y_1), \forall d_2 \in A(y_2).$$

Using the above mentioned brackets it is easy to see that the multivalued operator $A : D(A) \subset X \to 2^{X^*}$ is monotone if and only if

$$[A(y_1), y_1 - y_2]_- \geq [A(y_2), y_1 - y_2]_+ \quad \forall y_1, y_2 \in D(A).$$
In addition to the common monotony of multivalued maps we are interested in the following concepts (Figs. 2 and 3):

- **N-monotony**, namely
  \[
  [A(y_1), y_1 - y_2]_\pm \geq [A(y_2), y_1 - y_2]_\pm \quad \forall y_1, y_2 \in D(A);
  \]

- **V-monotony**, namely
  \[
  [A(y_1), y_1 - y_2]_+ \geq [A(y_2), y_1 - y_2]_+ \quad \forall y_1, y_2 \in D(A);
  \]

- **w-monotony**, namely
  \[
  [A(y_1), y_1 - y_2]_+ \geq [A(y_2), y_1 - y_2]_- \quad \forall y_1, y_2 \in D(A).
  \]
Definition 1. Let $D(A)$ be some subset. The multivalued map $A : D(A) \subset X \to 2^{X^*}$ is called:

- **Weakly $+$(-)-coercive**, if for each $f \in X^*$ there exists $R > 0$ such that
  $$[A(y), y]_{+(-)} \geq (f, y)_X \quad \forall y \in X \cap D(A) : \|y\|_X = R.$$  

- **$+$(-)-coercive**, if
  $$\frac{[A(y), y]_{+(-)}}{\|y\|_X} \to +\infty \quad \text{as} \quad \|y\|_X \to +\infty, \quad y \in D(A);$$

- **Uniformly $+$(-)-coercive** if for some $c > 0$
  $$\frac{[A(y), y]_{+(-)} - c\|A(y)\|_{+(-)}}{\|y\|_X} \to +\infty \quad \text{as} \quad \|y\|_X \to +\infty, \quad y \in D(A);$$

- **Bounded** if for any $L > 0$ there exists $l > 0$ such that $\|A(y)\|_{+(-)} \leq l \forall y \in D(A)$
  $$\|y\|_X \leq L;$$

- **Locally bounded**, if for an arbitrary fixed $y \in D(A)$ there exist constants $m > 0$ and $M > 0$ such that $\|A(\xi)\|_{+(-)} \leq M$ when $\|y - \xi\|_X \leq m, \xi \in D(A);$  

- **Finite-dimensionally locally bounded**, if for any finite-dimensional space $F \subset X$ the contraction of $A$ on $F \cap D(A)$ is locally bounded.

Definition 2. A strict multivalued map $A : X \xrightarrow{\quad} X^*$ is called:

- **Radial lower semicontinuous (r.l.s.c.)** if $\forall y, \xi \in X$
  $$\lim_{t \to 0^+} [A(y + t\xi), \xi]_{+(-)} \geq [A(y), \xi]_{+(-)};$$

- **Radial upper semicontinuous (r.u.s.c.)** if the real function
  $$[0, 1] \ni t \to [A(y + t\xi), \xi]_{+(-)}$$
  is upper semicontinuous at the point $t = 0$ for any $y, \xi \in X.$

- **Radial semicontinuous (r.s.c.)** if $\forall y, \xi \in X$
  $$\lim_{t \to 0^+} [A(y - t\xi), \xi]_{+(-)} \geq [A(y), \xi]_{+(-)};$$

- **Radial continuous (r.c.)** if the real function
  $$[0, 1] \ni t \to [A(y + t\xi), \xi]_{+(-)}$$
  is continuous at the point $t = 0$ from the right for any $y, \xi \in X;$
• (upper) Hemicontinuous (u.h.c.) if the function
\[ X \ni x \mapsto [A(x), y]_+ \]
is u.s.c. on \( X \) for any \( y \in X \);
• \( \lambda \)-pseudomonotone on \( X \) if for any sequence \( \{ y_n \}_{n \geq 0} \subset X \) such that \( y_n \rightharpoonup y_0 \) in \( X \) as \( n \to +\infty \) from the inequality
\[ \lim_{n \to \infty} \langle d_n, y_n - y_0 \rangle_X \leq 0, \quad (8) \]
where \( d_n \in \overline{\text{co}} A(y_n) \ \forall n \geq 1 \) the existence of subsequences \( \{ y_{n_k} \}_{k \geq 1} \) from \( \{ y_n \}_{n \geq 1} \) and \( \{ d_{n_k} \}_{k \geq 1} \) from \( \{ d_n \}_{n \geq 1} \) follows for which the next relation holds true:
\[ \lim_{k \to \infty} \langle d_{n_k}, y_{n_k} - w \rangle_X \geq [A(y_0), y_0 - w]_\mp \quad \forall w \in X ; \quad (9) \]
• \( \lambda_0 \)-pseudomonotone on \( X \), if for any sequence \( \{ y_n \}_{n \geq 0} \subset X \) such that \( y_n \rightharpoonup y_0 \) in \( X \), \( d_n \rightharpoonup d_0 \) in \( X^* \) as \( n \to +\infty \) where \( d_n \in \overline{\text{co}} A(y_n) \ \forall n \geq 1 \) from the inequality \( (8) \) the existence of subsequences \( \{ y_{n_k} \}_{k \geq 1} \) from \( \{ y_n \}_{n \geq 1} \) and \( \{ d_{n_k} \}_{k \geq 1} \) from \( \{ d_n \}_{n \geq 1} \) follows for which the relation \( (9) \) holds true.

The above mentioned multivalued map satisfies:
• Condition \((\kappa)+(\rightarrow)\) if for each bounded set \( D \) in \( X \) there exists \( c \in \mathbb{R} \) such that
\[ [A(v), v]_{\rightarrow} \geq -c \| v \|_X \quad \forall v \in D \setminus \{ 0 \} ; \]
• Condition \((\Pi)\) if for any \( k > 0 \), any bounded set \( B \subset X \), any \( y_0 \in X \) and for some selector \( d \in A \) for which
\[ \langle d(y), y - y_0 \rangle_X \leq k \quad \text{for all } y \in B , \quad (10) \]
there exists \( C > 0 \) such that
\[ \| d(y) \|_{X^*} \leq C \quad \text{for all } y \in B . \]

**Proposition 6.** \([ZMK10, Proposition 1.2.6]\) If a multivalued operator \( A : X \rightrightarrows X^* \) satisfies Condition \((\Pi)\) then it satisfies Condition \((\kappa)+(\rightarrow)\) as well.

During the investigation of evolution inclusions and variation inequalities, that describe mathematical models of nonlinear geophysical processes, we will use some properties of multimaps, represented in \([ZMK10]\) (Fig. 4).

**Corollary 1.** \([ZMK10, Corollary 1.2.2]\) Let \( \varphi : X \to \mathbb{R} \) be a convex lower semicontinuous functional such that
\[ \frac{\varphi(y)}{\| y \|_X} \rightharpoonup +\infty \quad \text{as } \| y \|_X \to \infty . \]
Then its subdifferential map

$$\partial \psi(y) = \{ p \in X^* | \langle p, \omega - y \rangle_X \leq \psi(\omega) - \psi(y) \quad \forall \omega \in X \neq 0, \quad y \in X$$

is $\partial$-coercive and hence $\psi$-coercive, uniformly $\psi$-coercive and uniformly $\partial$-coercive.

**Proposition 7.** [ZMK10, Proposition 1.2.30] Let a function $\psi : X \to \mathbb{R}$ be convex, lower semicontinuous on $X$. Then the multivalued map $B = \partial \psi : X \to C_v(X^*)$ is $\psi$-pseudomonotone on $X$ and it satisfies Condition (II).

Now let $X$ be a Banach space such that $X = X_1 \cap X_2$ where $X_1$, $X_2$ is the interpolation pair of reflexive Banach spaces [TRI78] which satisfies

$$X_1 \cap X_2 \text{ is dense in } X_1, X_2. \quad (11)$$

**Definition 3.** A pair of multivalued maps $A : X_1 \to 2^{X_1^*}$ and $B : X_2 \to 2^{X_2^*}$ is called $s$-mutually bounded if for each $M > 0$ and a bounded set $B \subset X$ there exists a constant $K(M) > 0$ such that from

$$\|y\|_X \leq M \quad \text{and} \quad \langle d_1(y), y \rangle_{X_1} + \langle d_2(y), y \rangle_{X_2} \leq M \quad \forall y \in B$$

it follows that

$$\|d_1(y)\|_{X_1^*} \leq K(M), \quad \text{or} \quad \|d_2(y)\|_{X_2^*} \leq K(M) \quad \forall y \in B$$

for some selectors $d_1 \in A$ and $d_2 \in B$.

**Remark 1.** [ZMK10, Remark 1.2.20] If one of the maps $A : X_1 \to 2^{X_1^*}$ or $B : X_2 \to 2^{X_2^*}$ is bounded then the pair $(A; B)$ is $s$-mutually bounded.
Lemma 1. [ZMK10, Lemma 1.2.9] Let $A : X_1 \Rightarrow X_1^*$ and $B : X_2 \Rightarrow X_2^*$ be some multivalued $+(-)$-coercive maps which satisfy Condition $(\kappa)$. Then the multivalued map $C := A + B : X \Rightarrow X^*$ is $+(-)$-coercive too.

Lemma 2. [ZMK10, Lemma 1.2.10] Let $A : X_1 \Rightarrow X_1^*$ and $B : X_2 \Rightarrow X_2^*$ be strict multivalued maps satisfying Condition $(\Pi)$. Then the pair $(A; B)$ is s-mutually bounded and the multivalued map $C := A + B : X \Rightarrow X^*$ satisfies Condition $(\Pi)$.

Now we consider subdifferential maps, that play an important role in the non-smooth analysis and the optimization theory [PSH80, AE84, DV81], in nonlinear boundary value problems for partial differential equations, the theory of control of the distributed systems [ZM99, LIO69], as well as the theory of differential games and mathematical economy [AF90, CHI97]. For basic properties of such maps we refer the reader to [AE84, DV81, IT79]. In [ZMK10] we generalized basic properties of subdifferentials and local subdifferentials known for Banach spaces to the case of Frechet spaces. We present some variations of obtained results, that we will use during the investigation of differential-operator inclusions and evolution variation inequalities for Earth Data Processing.

Let $U$ be a convex subset in $X$, $F : X \mapsto \mathbb{R} = \mathbb{R} \cup \{+\infty\}$ be a functional

$$\text{dom } F = \{x \in X | F(x) \neq +\infty\}.$$  

The set

$$\partial F(x_0; U) = \{p \in X^* | \langle p, x - x_0 \rangle_X \leq F(x) - F(x_0) \forall x \in U\}$$  

refers to a local subdifferential of a functional $F$ in a point $x_0 \in U$. Observe that $\partial F(x_0; U_1) \supseteq \partial F(x_0; U_2)$, if $U_1 \subset U_2$. In particular, $\partial F(x_0; X) = \partial F(x_0) \subset \partial F(x_0; U)$. The last set is called the subdifferential of $F$ at the point $x_0$ (Fig. 5).

**Proposition 8.** [AE84, p.191] Let $X$ be a Banach space. Then the norm $\| \cdot \|_X$ in $X$ is subdifferentiable functional and

$$\partial \| \cdot \|_X(x) = \{ p \in X^* | \langle p, x \rangle_X = \|x\|_X, \|p\|_{X^*} = 1\} \quad \forall x \in X.$$  

![Image](fig5.png)  

**Fig. 5** The local subdifferential map for $F(x) = \|x\| - 1$ as $x \in U = \mathbb{R}_+.$
In the case, when \( \alpha > 1 \)

\[
\partial \left( \frac{1}{\alpha} \| x \|_X^\alpha \right)(x) = \| \cdot \|^\alpha \partial \| \cdot \|_X(x)
\]

\[= \{ p \in X^* | \langle p, x \rangle_X = \| x \|_X^\alpha, \| p \|_{X^*} = \| x \|_{X^*}^{\alpha-1} \} \quad \forall x \in X.
\]

**Theorem 1.** [ZMK10, Theorem 1.3.3] Let \( U \) be a convex body in \( X \), \( F : X \to \mathbb{R} \cup \{ +\infty \} \) be a convex functional on \( U \) and a lower semicontinuous functional on \( \text{int}U \) (\( \text{int}U \subseteq \text{dom}F \)). Then for every \( x_0 \in \text{int}U \) and every \( h \in X \), the quantity

\[D_+ F(x_0; h) = \lim_{t\to 0^+} \frac{F(x_0 + th) - F(x_0)}{t}
\]

is finite and the following statements hold true:

(i) There exists a countercalibrated (cf. [RUD73]) convex absorbing neighborhood of zero \( \Theta \) \( (x_0 + \Theta \subseteq \text{int}U) \) such that for every \( h \in \Theta \)

\[F(x_0) - F(x_0 - h) \leq D_+ F(x_0; h) \leq F(x_0 + h) - F(x_0);
\]

(ii) The functional \( \text{int}U \times X \ni (x; h) \mapsto D_+ F(x; h) \) is upper semicontinuous.

(iii) The functional \( D_+ F(x_0; \cdot) : X \to \mathbb{R} \) is positively homogeneous and semiaffine for every \( x_0 \in \text{int}U \).

(iv) There exist a neighborhood \( O(h_0) \) and a constant \( c_1 > 0 \) such that for every \( x_0 \in \text{int}U \) and every \( h_0 \in X \),

\[|D_+ F(x_0; h) - D_+ F(x_0; h_0)| \leq c_1 d(h, h_0) \quad \text{for every } h \in O(h_0).
\]

**Definition 4.** A multivalued map \( A : X \rightrightarrows X^* \) is called:

(a) *-Upper semicontinuous (*-u.s.c.), if for any set \( B \) open in the \( \sigma(X^*, X) \) topology the set \( A_M^{-1}(B) = \{ x \in X | A(x) \subseteq B \} \) is open in \( X \).

(b) Upper hemicontinuous, if the function

\[X \ni x \mapsto [A(x), y]_+ = \sup_{d \in A(x)} \langle d, y \rangle_X
\]

is upper semicontinuous for each \( y \in X \).

Let us note that (b) follows from (a).

**Theorem 2.** [ZMK10, Theorem 1.3.4] Let \( U \) be a convex body and \( \text{int}U \subseteq \text{dom}F \), where \( F : X \to \mathbb{R} \) is a convex functional on \( U \) and a semicontinuous function on \( \text{int}U \). Then

(i) \( \partial F(x; U) \) is a nonempty convex compact set for every \( x \in \text{int}U \) in the \( \sigma(X^*, X) \) topology.

(ii) \( \partial F(\cdot; U) : U \rightrightarrows X^* \) is a monotone map (on \( U \)).
(iii) The map \( \text{int}U \ni x \mapsto \partial\varphi(x;U) \subset X^* \) is \(*\)-upper semicontinuous (on \( \text{int}U \)) and

\[
\left[ \partial\varphi(x_0;U), h \right]_+ = D_+\varphi(x_0; h) \quad \text{for all} \quad h \in X \text{ and } x_0 \in \text{int} U. \tag{14}
\]

**Theorem 3.** [ZMK10, Theorem 1.3.5] Let \( F : X \mapsto \mathbb{R} \) be a convex on \( U \) lower semicontinuous on \( \text{int}U \) functional. Then for each \( x_0 \in \text{intdom} \varphi \) the set \( \partial\varphi(x_0) \) is nonempty convex compact in \( \sigma(X^*; X) \)-topology, the map

\[
\text{intdom} \varphi \ni x \mapsto \partial\varphi(x) \subset X^*
\]

is \(*\)-u.s.c. and the following equality takes place

\[
\left[ \partial\varphi(x_0), u \right]_+ = D_+\varphi(x_0; u) \quad \forall u \in X. \tag{15}
\]

Some proofs are based on the following Proposition which is a generalization of
the Weierstrass Theorem on the case of locally bounded sets.

**Lemma 3.** [ZMK10, Lemma 1.4.3] Suppose \( W \) is locally convex space, \( W^* \) is its topologically adjoint, \( E \subset W^* \) is a set closed in the topology \( \tau(W^*; W) \).
\( \text{L} : E \rightarrow \mathring{R} = R \cup \{-\infty\} \) is its upper semicontinuous main functional in the topology \( \tau(W^*; W) \). Besides, let either the set \( E \) be equicontinuous in \( W^* \), or the following analogue of coercivity hold true: for an arbitrary set \( U \subset W^* \), which is not equicontinuous and \( \lambda \subset R \) there exists \( w_\lambda \in U \) such that \( \text{L}(w_\lambda) \leq \lambda \).

Then the functional \( \text{L} \) is upper bounded on \( E \) and reaches on \( E \) its upper boundary \( I \), and here the set \{\( w \in E | \text{L}(w) = I \} \) is compact in the topology \( \tau(W^*; W) \).

The role of the classical acute angle Lemma [ZM04] in demonstration of solvability for nonlinear operator equations with monotone coercive maps in finite-dimensional space is well-known. In [ZMK10, Sect. 1.4] the minimax inequalities are investigated. By using of this apparatus multivalued analogues of “acute angle Lemma” are proved.

**Corollary 2.** [ZMK10, Corollary 1.4.3] Suppose \( Y \) is finite-dimensional space, \( F : \overline{B}_r \rightarrow C_c(Y) \) are strictly u.s.c. maps where

\[
\overline{B}_r = \{ y \in Y | \| y \|_Y \leq r \}.
\]

If here

\[
[F(y), y]_+ \geq 0 \quad \forall y \in Y : \quad \| y \|_Y = r, \tag{16}
\]

then there exists \( \overline{x} \in \overline{B}_r \) for which \( \overline{0} \in F(\overline{x}) \).

By using this apparatus the new constructive solvability theorems for operator inclusions and variation inequalities are obtained in [ZMK10, Chap. 2]. More particular, suppose \( X \) is a reflexive Banach space, \( Y^* \) is a normalized space, \( U \subset Y^* \) is a nonempty subset, \( A : X \times U \rightarrow 2^{Y^*} \) is a multivalued map, \( f \in X^* \), \( u \in U \) are some fixed elements,
\[ K_{f,u} = \{ y \in X \mid f \in A(y,u) \}. \]

In [ZMK10, Chap. 2] it is investigated some properties of the set \( K_{f,u} \).

**Definition 5.** An element \( y \in X \) is called a weak solution of the inclusion \( f \in A(y,u) \) if

\[ [A(y,u),w]_+ \geq \langle f,w \rangle_X \quad \forall w \in X. \tag{17} \]

**Theorem 4.** [ZMK10, Theorem 2.2.1] Let for any \( u \in U \) \( A(\cdot, u) : X \to 2^{X^*} \) be a strict \( \lambda \)-pseudomonotone finite-dimensionally bounded map and for any \( f \in X^* \) there exists \( r > 0 \) such that

\[ [A(y,u) - f, y]_+ \geq 0 \quad \forall y \in \partial B_r \subset X. \]

Then \( \forall f \in X^*, u \in U \) there exists a weak solution of the operator inclusion \( f \in A(y,u) \).

**References**


Appendix A
Vortical Flow Pattern Past a Square Prism: Numerical Model and Control Algorithms

Abstract  A coupled Lagrangian–Eulerian numerical scheme for modeling the laminar flow of viscous incompressible fluid past a square prism at moderate Reynolds numbers is developed. The two-dimensional Navier–Stokes equations are solved with the vorticity–velocity formulation. The convection step is simulated by motion of Lagrangian vortex elements and diffusion of vorticity is calculated on the multi-layered adaptive grid. To reduce the dynamic loads on the body, the passive control techniques using special thin plates are proposed. The plates are installed either on the windward side of prism or in its wake. In the first case, the installation of a pair of symmetrical plates produces substantial decreasing the intensity of the vortex sheets separating in the windward corners of prism. In the second case, the wake symmetrization is achieved with the help of a long plate abutting upon the leeward surface. Both the ways bring narrowing of the wake and, as a result, decrease of the dynamic loads. With optimal parameters of the control system, the drag reduction is shown to decrease considerably.

A.1 Introduction

Let us consider some hydrodynamical application to differential-operator equations in infinite-dimensional spaces. We develop a coupled Larangian–Eulerian numerical scheme for modeling the laminar flow of viscous incompressible fluid past a square prism at moderate Reynolds numbers. Then we solve the two-dimensional Navier–Stokes equations with the vorticity–velocity formulation, that can be reduced to differential-operator equation. The convection step will be simulated by motion of Lagrangian vortex elements and diffusion of vorticity will be calculated on the multi-layered adaptive grid. To reduce the dynamic loads on the body, the passive control techniques using special thin plates will be proposed. The plates will be installed either on the windward side of prism or in its wake. In the first case, the installation of a pair of symmetrical plates produces substantial decreasing the intensity of the vortex sheets separating in the windward corners of prism. In the second case, the wake symmetrization is achieved with the help of a long plate abutting upon the leeward surface. Both the ways bring narrowing of the wake and,
as a result, decrease of the dynamic loads. With optimal parameters of the control system, the drag reduction will be shown to decrease considerably.

Hydrodynamic characteristics of a non-streamline body can be determined through vortex structure of the flow near it. Firstly they essentially depend on wall flow type, namely this flow is either separated or without separation. At large Reynolds numbers flows without separation are turbulent and a body drag can be determined through frequency and intensity of coherent vortex formations separations from the buffer zone of body boundary layer into the outward domain. In spite of small size these vortex structures essentially determine an energy interchange between the flow and the body. General pattern of flow past the body also depends on generation-diffusion correlation of vorticity in the wall area. In this case to construct control schemes providing the body drag decrease we should consider the dynamics of vortexes whose sizes are close either to the width of the buffer zone or to the boundary layer thickness. Great number of theoretical and experimental investigations [LN76, MO98, MIG62] show a principal possibility of the vortex structure control in the boundary flows. A number of schemes and constructions proposed for a boundary flow transformation, such as turbulence stimulators, vortex generators, vortex destructors, broaches, injectors of special flows etc. found their practical applications in aviation, naval technologies and hydraulic devices.

The the case of separated or detached flows past bodies of a non-streamline shape (namely the bodies with sharp edges or with large downstream pressure gradients over smooth surface areas). Such flows are accompanied by generation of vortex structures whose sizes are commensurate with geometrical parameters of the streamed body. Vortex generation causes irreversible energy consumption of the flow and brings unsymmetry into the patterns of flows past symmetrical constructions. In this case hydrodynamic characteristics of the body are determined by dynamics of the formed circulation flow. Instability and dynamic properties of a circulation zone are determined through motion and interaction of vortex structures in it.

It is well known that in a homogeneous incompressible fluid the vorticity is generated only by the body and be the domain boundaries where the fluid flows. An intensity of the vorticity generation depends on a local surface curvature of the body. On the smooth surface areas this intensity is much smaller than that of a flow past sharp edges. Hence sometimes it is enough to carry out dynamic analysis of vorticity separating from the body edges. Examples of such approach can be found in works devoted to the numerical analysis of a flow past a wing with a sharp trailing edge. Vortex sheets, which have been formed as a result of separation, under small perturbation decay into discrete vortexes, which, in their turn, after combining form into large circulation zones (Calvin–Gelmanhol instability). With the lapse of time as a result of viscous forces action diffusion processes start influencing upon the dynamics of these vortex formations, particularly, it is a viscous diffusion who causes the generation of a new vorticity on the smooth surface areas. Therefore considering a flow past a body we usually have several vorticity sources situated on the sharp edges and also on smooth surface areas. This sources are interacting both directly (mixing) and through hydrodynamic velocity field. The possibility of decreasing an
intensity of one vorticity source under the influence of the other attracts the practical interest. For example changing relative positions of vorticity sources through the body shape transformation we can largely decrease the intensity of vortexes generated by the flow past the body. In this case the body drag decreases and the level of non-stationary lateral forces reduces as well.

An artificial modification of the separated flow structure is one of the modern directions of the flow control theory. This theory is based on the development of algorithms for generation of large vortex formations with prescribed properties near the body [GG98, CLD94, COR96, SAV98]. Methods of artificial vortex generation can either passive (when our aim is to suppress vorticity generation processes that can be reached by modification of the construction shape) or active (with usage of control elements, for example, fluid withdrawal through slots or permeable surface areas, fluid injection, impurity of polymer additions into wall area through electric and magnetic fields action). The first class of these method does not require additional energy expenditure. But among the advantages of active schemes there is adaptivity, development and application possibilities for control algorithms together with their feedbacks when the intensity and the direction of an external influence on the flow depend on flow conditions.

Here we consider structure control schemes of the flow past of body non-streamline shape directed on decrease of hydrodynamic drug and construction vibration in the flow. It is important for increase of building reliability design, decrease of acoustic noises and power inputs caused by body motion in fluid. For control algorithm development understanding of generation and evolution processes as well as vortex dynamics in the flow past the body is important. The approbation of new algorithms can be carried out by the examples of the flow past relatively simple bodies. In view the analysis of flows past a square prism gives wide opportunities of approbation of control algorithm for flows past a body of non-streamline shape.

Investigation of the flow structure around the square prism is important in view of many factors: fundamental study of regularities of physical processes concerned with generation and interaction of separated circulation zones, obtaining patterns of flows past a body of non-streamline shape, solving practical engineering dynamics problems and reliability design problems for constructions exploitable in water or wind flows (flow past bridge bearings, tower buildings, elements of oceanographic equipment, offshore constructions).

In most experimental works devoted to the investigation of flows past the square prism an analysis was carried out at large Reynolds Numbers [BO82, VIC66, KNI90]. Exactly this range is important for the majority of engineering problems. The investigation results indicate that the flow pattern past the square prism and its hydrodynamical characteristics depend on Reynolds number far less than in the case of a round cylinder, and at \( \text{Re} \geq 10^3 \) these characteristics would hardly change a lot. This is a result by an existence of a fixed separation on the sharp edges causing approximate constancy of vortex separation frequency. Let us denote that at moderate Reynolds numbers \( 5 \cdot 10 \leq \text{Re} \leq 5 \cdot 10^3 \), the flow structure around the prism is
complicated enough. This fact results in certain changes of Strouhal number and of the square prism drag coefficient [OKA82].

First numerical investigation of flows past a square prisms are associated with the discrete-vortex method [NNT82]. Obtained there drag coefficients and frequencies of vortex separation are close to the experimental values at $Re \to \infty$. When Reynolds number is decreasing the mismatch between computed and experimental values ia increasing.

Numerical models of a flow past a square prism, considering viscous effects, are based on the complete Navier–Stokes equation system. Majority of such investigations concern with two-dimensional problems at small and moderate Reynolds numbers, One of the first and most complete investigations of this direction is the work [DM82]. The proposed algorithm combines elements of the grid method and the method of finite volumes. Taking into account limited capacity of the computation engineering, all computations was carried out with the help of crude mesh. Obtained data for hydrodynamic drag coefficient and Strouhal number, namely the number characterizing a frequency of vortex separations, differ within 10–20% from their experimental values.

In the work [MFR94] the study of a laminar flow past an oscillating square cylinder is carried out by the method of finite volumes. There are considered basic regimes occurring as a result of forced oscillations of the cylinder over its natural oscillations caused by vortex separation. In the work [SND98] this method had already been used for investigation of two-dimensional and three-dimensional flows past a square prism at moderate Reynolds numbers. Obtained results show that transition from two-dimensional regime up to three-dimensional one goes on in the following range of Reynolds numbers: from 150 to 200. At the same time it is shown that Strouhal number and body drag coefficient obtained through two-dimensional simulation are close enough to their experimental values.

Grid numerical algorithm applied in the work [ZSG94] for the analysis of flow past the square cylinder, revolving on longitudinal axis, is based on usage of Navier–Stokes equations with “vorticity-flow function” formulation. The advantage of this approach is absence of the pressure terms in the equations that enables to use explicit time computational schemes and avoids the choice boundary conditions for pressure.

In the majority of numerical simulation methods for viscous flows of incompressible fluid as an independent unknown magnitudes they take velocity and pressure. In view of numerical modeling, the form of Navier–Stokes equations that was proposed by Lighthill [LIG63] is more convenient. As an independent unknown magnitudes he considered velocity and vorticity. This gives an opportunity to separate a problem of fluid kinematics by applying the Bio–Savart law. Numerical models based on such approach can be reduced to the limit integral equation for the velocity. After that standing by themselves vorticity problems are considered which also can be reduced to limit integral equations [WW86]. The advantage of this approach is the following: when computing the velocity it is only vorticity distribution that is being taken into consideration. In the majority of hydromechanical problems the domain occupied by the whirling fluid is mach smaller than the domain of significant
velocity perturbations. Therefore the dimensions of the computation domain within the considered approach can be much smaller than in the case when required functions are velocity and pressure. Moreover in the problems of the external flow past a body, velocity boundary conditions of infinity are precisely fulfilled by analytical choice of Green’s function, entering into Bio-Savart equation. In some cases, for example in the case of a plane plate, the second boundary condition on the wall (non-percolation condition) can be similarly fulfilled.

Here for computation of the flow past the square cylinder and analysis some control schemes we propose generalized vortex algorithm. It is based on the complete Navier–Stokes equation system with the “vorticity–velocity formulation” and uses grids, Lagrangian vortex points and the discrete vortex method.

\section*{A.2 Problem Definition}

Let us consider a two-dimension laminar flow of viscous incompressible fluid past a square cylinder. Assuming critical parameters are the remote flow velocity $U_\infty$ and the length of the side of a square $a$, we obtain the following dimensionless Navier–Stokes equations:

$$
\frac{\partial V}{\partial t} + (V \cdot \nabla) V = -\nabla p + \frac{1}{\text{Re}} \nabla^2 V,
$$

\begin{align}
\nabla \cdot V &= 0, \quad (A.1) \\
\nabla \cdot V &= 0,
\end{align}

where $V(x, y, t)$ is the fluid velocity, $p(x, y, t)$ is the pressure, $\nu$ is the fluid viscosity, $\text{Re} = U_\infty a / \nu$.

Performing the operation \textit{rotation} with respect to each term of (A.1) and putting the vorticity $\omega = \nabla \times V$ in view of (A.2) we obtain the equation describing evolution and diffusion of vorticity in the considered domain. Particularly, for two-dimensional problems we have:

$$
\frac{\partial \omega}{\partial t} + (V \cdot \nabla) \omega = \frac{1}{\text{Re}} \Delta \omega. \quad (A.3)
$$

Equation (A.3) implies: if in the time point $t$ velocity and vorticity are given, then the vorticity distribution in the next time point $t + \Delta t$ can be found. Thereafter, under obtained values $\omega$, using Bio-Savart formula and taking into account boundary conditions on the body surface, we can find new velocity values in the domain. This computation cycle, which firstly was described by Lighthill [LIG63], is the foundation of the vortex method. The distinctive feature of numerical algorithms based on this cycle consists in the way how diffusion and vorticity convection are calculated, and also in different approaches to modeling of vorticity generation on the body wall.
To solve the diffusion problem for vorticity \( \omega \) it is necessary to fulfill boundary conditions of the body and boundary conditions of infinity. For velocity \( V(x, y, t) = V_n(x, y, t) + V_\tau(x, y, t) \) these are the standard non-percolation and adhesion conditions:

\[
V_n(x, y, t)|L = U_{n\, body}, \quad (A.4)
\]

\[
V_\tau(x, y, t)|L = U_{\tau\, body}, \quad (A.5)
\]

where \( U_{body} = U_{n\, body} + U_{\tau\, body} \) is the body velocity consisting of translation and rotation velocities in general case, \( L \) is the boundary of the body.

The choice of a boundary condition on the surface of the body \( L \) for the function \( \omega \) is a non-trivial problem. It is associated with the way one describes the vorticity generation on the body wall. Due to Lighthill’s method, which is effectively used in the modeling of discrete vortexes, the body surface is replaced by vortex sheet. In this case the vorticity value on the wall \( \omega_0 \) depends on intensity of the vortex sheet \( \gamma \). There are different approaches to finding a function \( \omega \). One of them is based on the fact that a jump of tangential velocity \( (V_\tau) \) in incompressible ideal fluid crossing the vortex sheet is equal to \( \gamma / 2 \). Then, under the adhesion condition, on the surface \( L \) the following relationship must be fulfilled:

\[
V_\tau^0 + \gamma / 2 = 0.
\]

Taking into account that \( \omega_0 = \gamma / h \), \( (h \) is a given short distance from the wall along the normal line or an appropriate sampling interval for computational grid associated with the body), we have:

\[
\omega_0 = -\frac{2V_\tau^0}{h} \bigg|_{L}. \quad (A.6)
\]

The velocity \( V_\tau \) in formula (A.6) is calculated directly on the wall.

In [WU76] the magnitude \( \omega_0 \) was determined using the expansion of tangential velocity into Taylor’s series near the body surface. In this case, taking for instance the horizontal wall, we have:

\[
\omega_0 = -\frac{2V_\tau(x, h/2)}{h} + \frac{\partial^2 V_\tau}{\partial y^2} \bigg|_{y=0} h/4 + O(h^2) + \ldots. \quad (A.7)
\]

If in formula (A.7) only terms containing first order infinitesimals are left, we obtain an expression which is an analogue of well-known Thompson’s formula in \( \omega - \psi \) model. Expressions (A.6), (A.7) are the examples of Dirichlet condition on the function \( \omega_0 \). In the work [KOU93] for vorticity flow Neumann condition is used. It should be noted that there is no rigorous mathematical substantiation of boundary conditions for the function \( \omega \) which would correlate the intensity of the vortex
sheet around the body with vorticity generated by its walls. The choice of boundary condition for the function $\omega$ essentially depends on the numerical method used for solving vortex transfer equation (A.3).

Let us consider an unbounded fluid flow. Then for fluid velocity perturbations caused by the body, the damping condition is satisfied:

$$V(x, y, t) \rightarrow U_\infty, \quad \text{if} \quad r = \sqrt{x^2 + y^2} \rightarrow \infty.$$ \hspace{1cm} (A.8)

The problem formulation is supplemented with initial conditions:

$$V(x, y, 0) = U_\infty(x, y),$$

$$\omega(x, y, 0) = \nabla \times U_\infty(x, y).$$ \hspace{1cm} (A.9)

### A.3 Numerical Algorithm

For modeling the fluid flow described by (A.3) with corresponding boundary and initial conditions we propose the generalized vortex method combining grids usage with Lagrangian vortex elements. This method relies on construction of numerical algorithms based on discrete-vortex approximations.

The configuration of the computation domain and axes related with the considered body are shown in Fig. A.1. Since the body surface is the only vortex source in the flow we may assume that an input flow is vortex-free:

$$\omega|_{AB} = 0, \quad V|_{AB} = U_\infty.$$ \hspace{1cm} (A.10)

![Fig. A.1 Computation domain](Fig_A_1.png)
On other boundaries of this domain we suppose [GG05]:

\[
\frac{\partial^2 \omega}{\partial y^2}\bigg|_{BC} = 0,
\]

\[
\frac{\partial^2 \omega}{\partial y^2}\bigg|_{AD} = 0,
\]

\[
\frac{\partial^2 \omega}{\partial x^2}\bigg|_{CD} = 0.
\] (A.11)

The boundary of the body \( L \) is simulated by continuous vortex sheet. Its intensity \( \gamma \) can be found using limit integral equations method. Taking into account the non-percolation condition (A.4) we obtain the following equation:

\[
\int_L \gamma(\mathbf{r}', t) \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} d l(\mathbf{r}') + \int_S \omega(\mathbf{r}', t) \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} d s(\mathbf{r}') = V_{\text{body}}
\] (A.12)

where \( \mathbf{r} = \mathbf{r}(t) \), \( \mathbf{r}' = \mathbf{r}'(t) \) are position vectors, \( G(\mathbf{r}, \mathbf{r}') \) is Green function, which for two-dimensional problem and unbounded fluid flow is of the form:

\[
G(\mathbf{r}, \mathbf{r}') = \frac{1}{2\pi i} \ln(|\mathbf{r} - \mathbf{r}'|).
\]

Moreover, the following theorem on constancy of circulation in the domain must take place:

\[
\int_L \gamma(\mathbf{r}', t) d l(\mathbf{r}') + \int_S \omega(\mathbf{r}', t) d s(\mathbf{r}') = 0.
\] (A.13)

The velocity field in the domain is defined due to Bio-Savart theorem:

\[
V(\mathbf{r}, t) = U_\infty + \int_L \gamma(\mathbf{r}', t) \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} d l(\mathbf{r}')
\]

\[
+ \int_S \omega(\mathbf{r}', t) \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n} d s(\mathbf{r}').
\] (A.14)

The vorticity field is approximated by the system of Lagrangian vortex elements, namely elements moving together with the fluid and circulating:

\[
\omega(\mathbf{r}, t) \approx \sum_k \Gamma_k f_\delta(\mathbf{r} - \mathbf{r}_k),
\] (A.15)

where \( \Gamma_k \), \( \mathbf{r}_k \) is a circulation and a position of \( k \)-th vortex, respectively, \( f(\mathbf{r} - \mathbf{r}_k) \) is a vortex function which is equal to delta function for ideal fluid, \( \delta \) is a radius
of vortex core (we assume that inside vortex core viscous effects are essential but outside the velocity field is potential).

If over the flow field we put the grid such that the vorticity \( \omega(x_k, y_k, t) \) is uniformly distributed in every its cell with the number \( k \), then for the circulation of the corresponding vortex element we have:

\[
\Gamma_k(t) = \omega(x_k, y_k, t) \Delta s_k,
\]

where \( \Delta s_k \) is grid element area.

Introduction of vortex core and the function \( f_\delta \) is one of the ways of vortex motion regularization, and it is widely used in modern vortex methods \([\text{NMDK99}, \text{CK00}]\). This approach allows to get rid of singularity in the vortex point, otherwise incredible velocities induced by adjoint vortices may occur. Instead of vortex points vortex blobs are considered with the core radius \( \delta \). For in-depth discussion concerning the choice of the function \( f_\delta \) and the radius \( \delta \) we refer to the monograph \([\text{CK00}]\). Here we use the second order Gaussian function \([\text{NMDK99}]\):

\[
f_\delta(\vec{r} - \vec{r}_k) = \frac{\exp\left(-\left(\vec{r} - \vec{r}_k\right)^2 / \delta^2\right)}{\pi \delta^2}.
\]

Core parameter essentially depends on the size \( h \) of an element of the orthogonal analytical grid:

\[
\delta = h^{0.9}.
\]

The process is being time sampled with the step \( \Delta t \). On each time step nonlinear \((\text{A.3})\) splits up on two equations, where the first one describes vorticity diffusion by means of viscous diffusion while the second one – by means of convection:

\[
\frac{\partial \omega}{\partial t} = \frac{1}{\text{Re}} \Delta \omega, \quad (\text{A.17})
\]

\[
\frac{\partial \omega}{\partial t} = -(V \cdot \nabla) \omega. \quad (\text{A.18})
\]

Let us note, that for numerical integrating of the equation system \((\text{A.17}), (\text{A.18})\) we use the explicit time integration scheme.

Over the flow field we consider an orthogonal analytical grid with the cell dimensions \( \Delta x \), \( \Delta y \). Then with every grid element we associate a vortex with intensity \( \Gamma_j = \omega_{ij} \Delta x \Delta y \).

As was mentioned before, the body surface is simulated by the vortex sheet. To find its intensity we apply the method of discrete vortices \([\text{GG05}]\) in the computational scheme. According to this method, sides of the square should be divided into equal vortex intervals, and each of them is replaced by a discrete vortex with circulation \( \Gamma_l^* \), that is equal to sheet intensity along the interval:

\[
\Gamma_l^* = \gamma^*(l) \Delta l, \quad l = 1, 2, \ldots, N,
\]
where $\Delta l$ is the length of the interval, $N$ is the number of vortexes, disposed along the boundary (attached).

The reference points in which integral equations (A.12), (A.13) hold true are disposed in the middle between neighboring vortexes. Then finding the intensity of the vortex sheet modeling the body boundary can be reduced to solving (at each time step) of $N$ linear algebraic equations with respect to unknown circulations of attached vortexes:

$$
\sum_{l=1}^{N} \Gamma^*_l (v^*_m)_{ml} = -2\pi U_\infty \sum_{i} \sum_{j} \Gamma_{ij} (v_n)_{mij}, \quad m = 1, 2, \ldots, N - 1,
$$

$$
\sum_{l=1}^{N} \Gamma^*_l = - \sum_{i} \sum_{j} \Gamma_{ij}, \quad (A.19)
$$

where $(v^*_m)_{ml}$ is a normal speed, inducible in $m$-th reference point by $l$-th attached vector, $(v_n)_{mij}$ is a normal speed in $m$-th reference point, inducible by vortexes disposed in mesh points outside the body boundary.

Solving system (A.19) and using formula (A.6), we can find vorticity $\omega$ on the body wall. To provide the fulfillment of the Kutt–Gukovsky condition in the sharp edges of the square, we put the grid along its surface in such a way that vortexes were in corner points. This vortexes are considered free, namely they are moving with the local velocity of the fluid. This approach was proposed by S.M. Belocerkovsky and was successfully used for computation of the flow past wings and bluff bodies applying discrete-vortex method [GG05].

Taking into account discretization of the flow field and of the body surface, formula (A.14) for computation of velocity components $u$, $v$ in the domain takes the following form:

$$
u(x, y) = U_\infty - \sum_{l=1}^{N} \frac{\Gamma^*_l}{2\pi} \frac{y - y_l}{(x - x_l)^2 + (y - y_l)^2}
	imes \left( 1 - \exp \left[ (x - x_l)^2 + (y - y_l)^2 \right] / \delta^2 \right)
\times \frac{\Gamma_{ij}}{2\pi} \frac{y - y_{ij}}{(x - x_{ij})^2 + (y - y_{ij})^2}
	imes \left( 1 - \exp \left[ (x - x_{ij})^2 + (y - y_{ij})^2 \right] / \delta^2 \right), \quad (A.20)
$$

$$
u(x, y) = \sum_{l=1}^{N} \frac{\Gamma^*_l}{2\pi} \frac{x - x_l}{(x - x_l)^2 + (y - y_l)^2}
$$

$u(x, y) = V_\infty - \sum_{l=1}^{N} \frac{\Gamma^*_l}{2\pi} \frac{y - y_l}{(x - x_l)^2 + (y - y_l)^2}
	imes \left( 1 - \exp \left[ (x - x_l)^2 + (y - y_l)^2 \right] / \delta^2 \right)
\times \frac{\Gamma_{ij}}{2\pi} \frac{x - x_{ij}}{(x - x_{ij})^2 + (y - y_{ij})^2}
	imes \left( 1 - \exp \left[ (x - x_{ij})^2 + (y - y_{ij})^2 \right] / \delta^2 \right), \quad (A.20)$
\[ \times \left(1 - \exp \left[ \frac{(x - x_i)^2 + (y - y_j)^2}{2} \right] \right) \]

\[ + \sum_i \sum_j \frac{I_{ij}^*}{2\pi} \frac{x - x_{ij}}{(x - x_{ij})^2 + (y - y_{ij})^2} \]

\[ \times \left(1 - \exp \left[ \frac{(x - x_{ij})^2 + (y - y_{ij})^2}{2} \right] \right). \]

To solve viscous diffusion equation (A.17) for second order space derivatives we use centered difference approximation on the analytical grid. The time derivative can be approximated using first order explicit scheme. Hence, if we know vortex values in mesh points at time point \( t \), its new values \( \omega_i^{t+\Delta t} \) can be found from the formula:

\[
\omega_i^{t+\Delta t} = \omega_i^t + \frac{\Delta t}{\text{Re}} \left( \frac{\omega_{i+1}^t - 2\omega_i^t - \omega_{i-1}^t}{\Delta x^2} + \frac{\omega_{j+1}^t - 2\omega_j^t - \omega_{j-1}^t}{\Delta y^2} \right).
\]  

(A.21)

Let us note, that applying formula (A.21) results in calculating errors which are the source of artificial viscosity. This circuit viscosity depends on discretization steps, both time and space ones. It is related, specifically, with negligible diffusion on the first grid layer adjoining wall along the surface normal. To reduce the error and circuit viscosity there can be used the exact solution of (A.17), which is of the form [GG05]:

\[
\omega^{t+\Delta t}(r) = \int_S \omega^t(\vec{r}^{'}) \ G(\vec{r}, \vec{r}^{'}) \ d\vec{r}'.
\]  

(A.22)

where \( G(\vec{r}, \vec{r}^{'}) \) is the Green function:

\[
G(\vec{r}, \vec{r}^{'}) = \frac{\text{Re}}{4\pi \Delta t} \exp \left[ -\frac{1}{4\Delta t} \left( \vec{r} - \vec{r}^{'} \right)^2 \right].
\]

Mathematical aspects of the scheme and problems arising when accounting a boundary effect on the diffusion process are considered in [BP86].

Approach based on the Green function (A.22) is more explicit though requires significant computer resources. Effectiveness of application of formulas (A.21), (A.22) depends on Reynolds number. Formula (A.21) is more effective for small Reynolds numbers, while the greater Reynolds numbers are the more explicit the second scheme is.

Equation (A.18) coincides with the corresponding equation describing vortex motion in the ideal incompressible fluid, where vortices move together with fluid elements and their intensity does not change. Therefore instead of (A.18) we may consider the equation describing the motion of vortex elements \( x_i, y_j \):
\[
\frac{dx_v}{dt} = u_v, \\
\frac{dy_v}{dt} = v_v, \quad (A.23)
\]

This approach is often used in traditional vortex algorithms which \cite{CK00} require carrying out complex procedure of regridding, namely redistribution of circulation of free vortex on the mesh points. Let us note, that numerical integrating of (A.23) and application of approximate “regridding”–algorithms considerably increases circuit viscosity of computational scheme.

Here to integrate (A.18) we propose to consider the flow field as a collection of discrete volumes, for each of which the vortex conservation law holds true:

\[
\int_{\Delta Q} \frac{\partial \omega}{\partial t} \, dq = - \int_{\Delta S} \omega (\nabla \cdot \vec{n}) \, dS, \quad (A.24)
\]

where \( \Delta Q \) is a discrete volume, on which flow field is divided, \( \Delta S, \vec{n} \) is a surface of this volume and its outer normal. To each discrete volume the mesh point \((x_{ij}, y_{ij})\) with vorticity \(\omega_{ij}\) is related (Fig. A.2). In view of (A.24), the vorticity convection of this grid element in the given time point \(t\) is described by the following expression:

\[
\frac{\Delta \omega_{ij}}{\Delta t} \, \Delta x \Delta y \approx \omega_{i-1,j} \, u_{i-1,j} \Delta y + \omega_{i,j-1} v_{ij-1} \Delta x - \omega_{i+1,j} u_{i+1,j} \Delta y - \omega_{ij} v_{ij} \Delta x.
\]

Hence we obtain the circulation of the considered vortex element in the next time point \(t + \Delta t\):

\[
\Gamma_{ij}^{t+\Delta t} = \Gamma_{ij}^t + \left( \omega_{i-1,j} \, u_{i-1,j} \Delta y + \omega_{i,j-1} v_{ij-1} \Delta x - \omega_{i+1,j} u_{i+1,j} \Delta y - \omega_{ij} v_{ij} \Delta x \right) \Delta t. \quad (A.25)
\]
Velocity components of the vortexes \( u, v \) can be found from expressions (A.20). It should be noted that the grid put over the flow field is adaptive, namely in each time point we consider only the mesh points with nonzero vorticity. This feature makes it possible to optimize calculations of velocity field.

Now let us consider the whole the computation algorithm in each time point:

1. Having known the vorticity values in the domain on the previous step, from the system of algebraic equations (A.19) we obtain the intensity of the attached vortex sheet and the intensity of vortexes on the sharp edges of the boundary.
2. From boundary condition (A.6) we obtain vorticity generated by the body surface.
3. Using boundary conditions (A.10), (A.11), we compute vorticity value on the boundaries of rated domain.
4. Using numerical integration of diffusion equation by formula (A.21) we calculate intermediate vorticity values \( \omega_{ij}^t \) in the inner points of flow area.
5. Taking into account the obtained vorticity distribution we correct the intensity of attached vortex sheet in such a way that surface non-percolating condition was fulfilled.
6. Having obtained the vorticity field we define the velocity field in the domain using formulas (A.20).
7. Using formula (A.25) we compute the convection of vortex elements. Obtained values \( F_{ij, \Delta t} \) in their turn define new values of vorticity \( \omega_{ij}^{t+\Delta t} \).

### A.4 Computation of Hydrodynamic Loads

To compute the hydrodynamic force acting on the body which is moving in the homogeneous fluid, we can use the theorem of impulses:

\[
\vec{F} = -\frac{d}{dt} \int_S \vec{V} \, dr.
\]

(A.26)

where \( \vec{F} = (F_x, F_y) \), \( S \) is a fluid-filled domain, \( \vec{V}, \vec{r} \) \((x, y)\) is a velocity and a position vector of a fluid element, respectively.

From the continuity equation and from the definition of vorticity \( \omega \) we obtain the following formula for fluid velocity:

\[
\vec{V} = (\vec{r} \times \vec{\omega}) + \nabla \cdot (\vec{r} \cdot \vec{V}) - \nabla(\vec{r} \cdot \vec{V}).
\]

(A.27)

Setting (A.27) in (A.26) and transforming integrals we obtain the force expression which depends only on the vorticity field characteristics [WU81, GG05]:

\[
\vec{F} = -\frac{d}{dt} \int_S \omega \times \vec{r} \, d\vec{r}.
\]

(A.28)
This implies formulas for computation of resistance and lifting force:

\[
F_x = -\frac{d}{dt} \int_S \omega y ds, \\
\text{[3mm]} F_y = \frac{d}{dt} \int_S \omega x ds.
\]

(A.29)

Note that force component \( F_x \) in (A.29) includes form resistance as well as friction resistance.

Since the vorticity field is considered as superposition of vortex elements with intensities \( \Gamma_{ij} \) and coordinates \((x_{ij}, y_{ij})\), we have formulas (A.29) in discrete form:

\[
F_x = -\sum_i \sum_j \left( \frac{d\Gamma_{ij}}{dt} y_{ij} + \Gamma_{ij} v_{ij} \right), \\
\text{[5mm]} F_y = \sum_i \sum_j \left( \frac{d\Gamma_{ij}}{dt} x_{ij} + \Gamma_{ij} u_{ij} \right). 
\]

(A.30)

Dimensionless coefficients of hydrodynamic forces are defined in the following way:

\[
C_x = \frac{2F_x}{U_\infty^2 a}, \quad C_y = \frac{2F_y}{U_\infty^2 a}. 
\]

(A.31)

### A.5 Approbation of Numerical Scheme

Testing of the developed algorithm was carried out in several directions: analysis of numerical diffusion of computational scheme; accuracy evaluation of the algorithm with respect to description of vorticity generation on the body surface, error analysis when defining hydrodynamic forces acting on a streamline body.

#### A.5.1 Evolution of Vortex Point in Unbounded Flow

Suppose that in an unbounded domain there is a vortex point with intensity \( \Gamma_0 \). Its vorticity diffusion can be calculated by formulas (A.21), (A.23). Obtained vorticity distribution we compare with the known exact solution of Navier–Stokes equation which describes the development of two-dimensional vortex with the initial circulation \( \Gamma_0 \) in the viscous fluid [GG05]:

\[
\omega(x, y, t) = \frac{Re}{4\pi t} \exp \left( -\frac{r^2 Re}{4t} \right),
\]

(A.32)

where \( Re = \Gamma_0/v, \ r = \sqrt{x^2 + y^2}, \ v \) is fluid viscosity.
In Fig. A.3 we can see time dependencies of vortex radius \( r_v \), calculated on analytical grid \( \Delta x = \Delta y = 0.01 \) with the time step \( \Delta t = 0.0025 \) (markers), and the exact values \( r_v \) obtained from formula (A.32) (total line). The vortex radius is obtained from the relationship:

\[
\omega |_{r>r_v} < 10^{-5}
\]

From Fig. A.3 it follows that in the case \( \text{Re} = 10^2 \) numerical diffusion is insignificant, and for \( \text{Re} = 10^3 \) its value is within 10\%. Increasing of numerical diffusion at large Reynolds numbers is essentially related with discretization parameters misfit: the steps \( \Delta x, \Delta y \) are too big for such Reynolds number. The best result is attained if \( \Delta x, \Delta y \approx 1/\text{Re} \). Comparison of curves in Fig. A.3 indicates that the numerical algorithm is convergent at moderate Reynolds numbers.

The choice of parameter value \( \Delta t \) is associated with the dimensions of grid elements and with local flow velocity. To provide the stability of the algorithm, vortex displacement within a time step must not exceed the minimal dimensions of a grid element \( \Delta x, \Delta y \). In most cases it is enough that the step \( \Delta t \) was less by an order of the dimensions of grid elements.

### A.5.2 Flow Past a Flat Plate

For accuracy evaluation of developed numerical algorithm describing vorticity generation process on the body surface we solved the problem of longitudinal flow past a flat plate. Characteristic parameters of this problem are the flow velocity \( U_\infty \) and the length of the plate \( L \), such that: \( \text{Re}_L = U_\infty L / \nu \), \( \bar{x} = x / L \), \( \bar{y} = y / L \). Calculation of vorticity generation based on the boundary condition (A.6). For modeling of vorticity diffusion formulas (A.21), (A.23) was used. Calculation results was compared with Blazius solution, which comes from boundary-layer theory [GG05]. In Figs. A.4 and A.5 we can see the stationary profile of longitudinal velocity \( \bar{u} = u / U_\infty \) for the middle of the plate and the dimensionless friction factor

![Fig. A.3 Vortex diffusion in unbounded domain. Comparison of calculation results open circle for vector \( r_v \), with the exact solution solid line : 1, \( \text{Re} = 10^2 \); 2, \( \text{Re} = 10^3 \)](image)
Fig. A.4 Longitudinal velocity profile in the attached layer for the middle of the plate at $\text{Re}_L = 10^3$: solid line Blazius solution, filled circle computations

Fig. A.5 Plate’s longitude coordinate dependence of the friction coefficient at $\text{Re}_L = 10^3$: solid line Blazius solution, dashed line computations

$$
\overline{C}_f = \frac{2}{\text{Re}_L} \frac{\partial u}{\partial y}|_{\text{wall}} = -\frac{2}{\text{Re}_L} \omega|_{\text{wall}} \quad \text{(A.33)}
$$

on the plate, numerically obtained and obtained due to Blazius solution at $\text{Re}_L = 10^3$. The comparison of these results indicates of high accuracy of considered scheme for modeling velocity and vorticity fields and for definition of hydrodynamic friction drag force of the plate.

### A.5.3 Flow Past Square Prism

Developed numerical algorithm was used for modeling of two-dimensional laminar flow past a square prism at moderate Reynolds numbers. Carried out computations revealed some regularities of separated flow forming as well as supplemented the approbation of the numerical scheme by comparing obtained characteristics with well-known experimental data and similar numerical results of other researchers.

Computations was carried out in the range of Reynolds numbers from $\text{Re} = 70$ to $\text{Re} = 5 \times 10^3$. Further increase of Reynolds number needs considerable computational forces, due to decreasing dimensions of grid elements, as well as application of turbulence models. The grid which was used for calculations had square elements
and consisted of tree levels (Fig. A.6) with different densities. On the first level that is adjacent to the body, the dimensions of grid elements is associated with the number $N$ of attached vortexes located along the prism walls (on the square side). On the next levels the dimensions of elements enlarge (double). Most of computations are done at $N = 100$.

In Fig. A.7 we see the computed instantaneous vorticity distributions in the wake past the square cylinder at different Reynolds numbers and at $\tau = 25(\tau = tU_\infty/a)$. This results indicate that the pattern of flow past the square cylinder considerably depends on $Re$. At $Re \leq 100$ directly adjacent to the body circulation zones are of prolate form (Fig. A.7a). Intensity of vortex structures separating from this zones is comparatively small. Structures interactions in the wake lead to forming of the regular vortex street. When the Reynolds number increases the length of circulation zones decreases while the intensity of separating vortex beams proportionally goes up. Respectively vortex interaction in the wake amplifies and the width of the wake considerably increases. Since there is more vortexes entering the wake, the cylinder drag increases as well. At $Re \geq 1,000$ the origin of the vortex sheet comes nearer to the body (Fig. A.7d–e). Increase of vortex circulation results in intensification of their interaction which in its turn brings increase of the street width as well as considerable deformations of separated vortex structures (up to their breakdown).

Mentioned features of the flow correlate with dependencies of instantaneous drag coefficient $C_x$ and lifting force coefficient $C_y$ on Reynolds numbers (Fig. A.8). Cited diagrams indicate that after short transient time phase in the wake past the body at $\tau > 10$ the structure of the flow approximates a periodic one that is characterized by Strouhal number $St = fa/U_\infty$, where $f$ is a frequency of vortex separation. At $Re > 1,000$ an additional mode appears. This fact is corroborated by the vortex distribution data in the wake (Fig. A.7e).
Fig. A.7  Vorticity distribution in the wake past the square cylinder at different Reynolds numbers, $\tau = 25$
Fig. A.8  Time dependence of drag coefficient $C_x$, 1 and of lifting force $C_y$, 2 of the square prism at different Reynolds numbers

Fig. A.9  Dependence of a Strouhal number on a Reynolds number for the flow past the square prism and comparison of the obtained results with the experimental data; [OKA82] filled circle computations, open circle, open triangle, ... experiment

In Fig. A.9 the computation of Strouhal numbers are compared with the data from the work [OKA82] that is notable for broad experimental investigations of the flow past square prism at moderate Reynolds numbers. This comparative diagram indicates that the introduced numerical scheme high-precisely describes forming of the vortex wake past the body in the range of moderate Reynolds numbers.
In Fig. A.10 we see computed by formulas (A.31) dependencies of the period average drag coefficient $\overline{C}_x$ and the lifting force oscillation amplitude $C_{y}^{max}$ of the square prism on Reynolds numbers. Let us denote that dependencies of the drag coefficient $\overline{C}_x$ and the frequency of vortex separation (of the number St) on Reynolds number are correlating; increase of Strouhal number up to $St = 1.45$ at $Re = 250$ is implied by narrowing of the wake directly after the square. This fact in its turn leads to the drag reduction (Fig. A.10a). Narrowing of the wake is essentially associated with a change of the flow pattern around the prism. At $Re > 200$ the vortical circulation zone arising near a side wall of the prism decreases in both crosswise and lengthwise directions (though, experiments indicate that the reattachment of the circulation flow to the walls in the case of a square prism does not take place). Further increase of $\overline{C}_x$ is implied by wake expansion due to decrease of viscous diffusion and of vortex circulation in the wake. The obtained data (Fig. A.10) for the magnitudes $\overline{C}_x, C_{y}^{max}$ is close to the relevant results introduced in the works [DM82, MFR94], where computations were based on finite volume method. Experimental investigations of dynamic loads on the square cylinder substantially deal
with large Reynolds numbers \( \text{Re} \geq 5 \cdot 10^3 \). In this case the flow is essentially tree-dimensional and its characteristics depend on the rate of flow turbulence, velocity profile and also on parameters of experimental facility. This fact explains significant variation of values \( C_x \) obtained in different experiments.

In general the obtained results corroborate that the introduced numerical scheme is an effective tool for modeling of the viscous flow and it can be used for computation of flows past the bodies of complex configuration.

### A.6 Algorithms for Flow Past a Square Cylinder Structure Control

Theoretical and experimental investigations of laminar flow past a square (namely two-dimensional case of flow past a square prism) at moderate Reynolds numbers indicated that wake structure and hydrodynamic drag coefficient \( C_x \) and lifting force coefficient \( C_y \) essentially depend on dimensions and interaction character of separated circulation zones, forming after the body.

Two following cases are qualitatively different: – when a separated zone over the side wall is local (bounded); – when a separated circulation zone generated on the windward corner of the prism is unclosed while its leeward corner is situated in the middle of this zone. In the first case the interaction between primary and quarter zones is weak and the wake progress is determined mostly by separation on the leeward corner. In the second case it is separation on the windward corner that is of most importance for the wake progress. The following algorithm for structure control of the flow past a square prism is based on physical properties of interaction between separated zones generated on windward and leeward corners of the prism.

The control objective is drag reduction of dynamic lateral forces producing hydroelastic oscillation of the flow structure. The strategy of such control is reduction of total vortex intensity, reduction of vortex flow past the body and minimization of the wake size. We propose to achieve an improvement of the flow structure around the square prism through artificial generation of special local separated circulation zones (standing vortexes) near it. Physical experiments carried out in Institute of Hydromechanics of National Academy of Sciences of Ukraine with non-streamline bodies demonstrated an efficiency of this approach for flow structure change as well as for improvement of hydrodynamic characteristics [KR95]. To achieve the necessary positive effect an artificial separated zone must be stable with respect to external perturbations. Otherwise these perturbations may cause vortex emission into the flow. Moreover, a separated zone must be controllable (namely when its dimensions change with respect to the change of the flow conditions). In the optimal case the structure control system must have a feedback.
A.6.1 Two Symmetric Plates on the Windward Side of the Body

Here we consider the control technique of the flow past a square prism using special thin plates which are installed on the windward side of the prism (Fig. A.11). In the space between such plate and the corner of the prism a local separated zone is forming. Experiments indicate that the control objective is achieved when the flow line being separated from the plate edge $A$ is attaching gradually to the corner $B$.

To study the flow past the body with control plates (Fig. A.11) we propose a few theoretical models. Certain general properties of detached flow can be described by the simplified model where the fluid supposed ideal and the circulation zone is simulated by one point vortex. For the justification of such model, it is associated with presence of general dynamic characteristics of separated zones which are the same for laminar, turbulent and potential flows (where viscous effects are insignificant). The analysis of dynamics of the point vortex describing separated circulation flow gives an information about topological characteristics of the flow: presence, disposition and type of critical points, their stability and reaction with respect to external perturbations [GG98, GG96].

Taking in view symmetry and stability of the flow in front of the body we consider only its upper part $y > 0$ (Fig. A.11). As before the boundary of the body is simulated by continuous vortex sheet which in its turn is replaced by a system of discrete vortexes in the numerical scheme. Then the complex potential of the flow is of the form:

$$\Phi(z) = U_\infty z + \frac{1}{2\pi i} \sum_{k=1}^{N} \Gamma_k \ln \frac{z-z_k}{z-z_0} + \frac{1}{2\pi i} \Gamma_0 \ln \frac{z-z_0}{z-z_0},$$

(A.34)

where $\Gamma_k$ and $z_k = x_k + iy_k$ is a circulation and complex coordinate of $k$-th attached (situated on the boundary) discrete vortex, respectively, $\Gamma_0$, $z_0 = x_0 + iy_0$ are parameters of a standing (immobile) vortex describing the circulation flow, $N$ is a number of attached vortexes.

![Fig. A.11 Configuration of the body with control plates](image_url)
The problem is to find the parameters of the standing vortex \( \Gamma_0 = \Gamma_0 / U_\infty a \), \( x_0 = x_0 / a \), \( y_0 = y_0 / a \) and parameters of the control plate \( \bar{r} = r / a \bar{t} = l / a \), (see Fig. A.11) such that there is a lack of vorticity generation in the points \( A \) and \( B \) (further we omit the dash indicating that the magnitude is dimensionless).

For solving such problem we have four nonlinear equations. The first two of them follow from vortex immobility condition:

\[
\frac{d\Phi}{dz} \bigg|_{z=z_0} = 0
\]  

(A.35)

or

\[
v_x(z_0) = 0, \quad v_y(z_0) = 0.
\]

Another two equations describe the Kutt–Gukovsky condition about velocity boundedness in the corner points \( A \) and \( B \). The flow is optimal if

\[
\Gamma_k \big|_A = 0, \quad \Gamma_k \big|_B = 0,
\]

(A.36)

where \( \Gamma_k \big|_A \) and \( \Gamma_k \big|_B \) are circulations of discrete vortexes, situated in the corner points \( A \) and \( B \).

Taking in view the expression for the potential (A.34), the conditions (A.36) can be reduced to the following form:

\[
v_x(z_0) - i v_y(z_0) = U_\infty + \frac{\Gamma_0}{4\pi y_0} + \frac{1}{2\pi i} \sum_{k=1}^{N} \Gamma_k \left( \frac{1}{z_0 - z_k} - \frac{1}{z_0 - \bar{z}_k} \right) = 0.
\]  

(A.37)

Unknown circulations of the attached vortexes \( \Gamma_k \) can be found from the system of linear equations that follows from non-percolation condition on the body surface and is similar with the system (A.19) where instead of the collection of vortexes we consider only one circulation vortex \( \Gamma_0 \).

The equations for \( \Gamma_0, x_0, y_0, r \) are essentially nonlinear. To solve them we used Broyden’s numerical algorithm. Computations indicated that given one of control plate characteristics, for instance, the length of the plate \( l \), the problem (A.35)–(A.36) has a unique solution; in the space between the plate and the prism side at \( 0.15 \leq l \leq 0.65 \) there is a point with coordinates \( x_0, \quad y_0 \) where the vortex with circulation \( \Gamma_0 \) is equilibrion (namely it is immobile or standing); exactly such vortex provides null vortex generation in the corner points \( A \) and \( B \). The numerical behavior analysis of this vortex in the neighborhood of the equilibrium position showed that small flow perturbations result in precessional movement of the vortex about the point \( x_0, \quad y_0 \). In view of dynamic system theory it means that such critical point is an elliptic one and the corresponding vortex is stable. Figure A.12 illustrates dependencies of the standing vortex coordinates \( x_0, \quad y_0 \), circulation \( \Gamma_0 \) and the parameter \( r \) on the plate length \( l \). These dependencies are close to linear ones. From the graph \( r(l) \) we can define an optimal configuration of the control plates. The pattern of flow lines in front of the long body for the case \( l = 0.3, \quad r = 0.22 \) is shown in Fig. A.13.
Fig. A.12 Dependencies of standing vortex circulation, its disposition and the parameter $r$ on the plate length $l$: 1. $x_0(l)$; 2. $y_0(l)$; 3. $\Gamma_0(l)$; 4. $r(l)$

Fig. A.13 The pattern of flow lines with forming of the standing vortex in front of the long body with the control plates on the windward side

Computations carried out with respect to the simplified model, namely when the separated circulation flow is described by one point vortex, demonstrated principal possibility of existence of a standing (immobile) vortex in front of the body with special (control) thin plates on its windward side, and also this computation make it possible to define optimal parameters of the plates. For this way obtained optimal configuration of a square prism with control plates the flow past this prism was simulated basing on the complete Navier–Stokes equation system due to the scheme defined in the Sect. A.2. Figure A.14 illustrates instantaneous vorticity distribution in the wake past the square cylinder with control plates at $Re = 500$, when the characteristics of the plates are close to optimal ones: $l = 0.2, r = 0.16$.

Comparison of computation results shown in Figs. A.14 and A.7c indicates that of the wake past the prism changes after the installation of the plates. This change is expressed in:

- Narrowing of the vortex region both near the body and in the wake.
- Increasing of frequency of vortex separation and, respectively, extension of the wake; Strouhal number characterizing this process increases from $St = 0.125$ for an ordinary square prism up to $St = 0.143$ for a prism with control plates.
- Decreasing intensity of vortexes in the wake.
Fig. A.14 Vorticity distribution in the wake of the square cylinder with control plates in the optimal regime ($l = 0.2, r = 0.16$) at $Re = 500, \tau = 25$

Fig. A.15 Time dependencies of drag coefficient $C_x$ (a) and lifting force coefficient $C_y$ (b) for the square cylinder at $Re = 500$: 1 with control, 2 with optimal control ($l = 0.2, r = 0.16$)

Specified changes have a positive effect on hydrodynamic characteristics of the prism (Fig. A.15, $Re = 500$). The value of period average drag coefficient $C_x$ at moderate Reynolds numbers decreases from 2.15 (for an “ordinary” square prism) down to $C_x \approx 1.2$, that is about 55% of the previous value (Fig. A.15a). The lifting (or lateral) force oscillation amplitude respectively decreases from $C_y^{max} = 2$ down to $C_y^{max} = 0.8$ (Fig. A.15b). If the body is elastically fixed in the flow then the amplitude of the body oscillations caused by vortex separation substantially decreases.

Importance of parameter choice for control plates is demonstrated in Fig. A.16, where we present dependencies of the coefficients $C_x(\tau), C_y(\tau)$ at different plate–prism corner distances. Graphs $C_x(\tau), C_y(\tau)$, and the vorticity distribution in the wake (Fig. A.17) indicate occurrence of additional modes caused by vortex deformations in the wake.

Conclusions concerning optimal flow structure can be generalized over the range of large Reynolds numbers. This fact is supplemented by the results of numerical
Fig. A.16  Time dependencies of drag coefficient $C_d$ (a) and lifting force coefficient $C_y$ (b) for the square cylinder at different parameters of control plates:  
1 $l = 0.2, r = 0.16$,  
2 $l = 0.2, r = 0.4$,  
3 $l = 0.2, r = 0.075$,  
Re = 500

Fig. A.17  Vorticity distribution in the wake of the square cylinder with control plates in non-optimal regime ($l = 0.2, r = 0.075$) at Re = 500, $\tau = 25$

modeling for detached flow past a prism and a prism with plates based on the method of discrete vortexes. We supposed that from the corner points vortex sheets come down into the flow.

The computation results indicate three typical regimes of a flow past the body with control plates. If the plate is installed far from the corner then the vortex sheet and the corresponding flow line which has descended from the plate edge lie on the windward side of the square (Fig.A.18c). In this case formed circulation zone becomes weak and is of little influence upon the flow development around the body. The flow attaches to the body at the point lying substantially lower that the corner $B$. The vortex generated in front of the body weakly influence on vortex generation processes in the corner point $B$ (on the edge of the body), and hence it cannot prevent a global flow separation in the neighborhood of this corner, and in its this turn causes a development of an intense vortex zone over the body wall.
The second type of the flow occurs when the control plate is very close to the corner (Fig. A.18b). In this case the flow does not attach to the body, and the detached circulation zone does not localize. On the contrary, an intense detached zone is developing, as it had already been observed in the case of a body without plates. Let us note also, that approximated one-vortex model, describing a circulation flow through the behavior of the one vortex, did not show up any standing vortex in the space between the plate and the corner. When the plate is very close to the corner $B$ the separation zone descending from the plate edge $A$ overlays on the flow generated by the prism edge. The flow in the neighborhood of the corner $B$ becomes very unstable. In this situation the body characteristics become worse compared with the case without the controls.

In the optimal case the flow line descending from the plate edge is gradually attaching to the body surface near the corner $B$ (Fig. A.18a). Further this attached flow without any relevant perturbations moving along the surface. Vorticity generation on the corner $B$ is suppressed by influence of the circulation zone in front of the body. In this case we can see the decrease of vortex intensity in the flow round the corner $B$ that causes substantial drag decrease. It has to be mentioned that the optimal control plate parameters computed by applying each of the following schemes: the “viscous” model, the method of discrete vortices and the model of the standing vortex, are almost coincident.

### A.6.2 Stabilization of a Vortex Flow Past the Body with a Separating Plate

Substantial influence on the body drag has a structure of a flow in the near wake. It depends on intensity of vortex formation and floe pattern in the wake. Two following situations are possible:

- The flow past the body is stable; here two symmetrical circulation zones are forming; the process of whirling liquid carrying away into encircling flow is insignificant, hence the body drag is relatively small as well.
- Circulation zones past the body are unstable with respect to small perturbations; from their interaction area attaching directly to the leeward side of the body large beams of whirling fluid periodically join to the flow; hydrodynamic forces in this case are much greater that in the previous case.

A stable symmetrical flow pattern in the wake occurs in the experiments only at small Reynolds numbers $Re < 50$, when perturbations are suppressed by viscous force action. The stability analysis of the flow in the wake showed that non-symmetrical (with respect to the axis $Ox$ – Fig. A.1 in our case) perturbations are most dangerous (destabilizing). To suppress such perturbations and stabilize the flow is possible using a separating plate situated behind the body (Fig. A.19). To find out regularities of vortex structure forming for a flow past a square prism with a plate we simulated the flow using numerical scheme defined in the Sect. A.2.
We considered laminar flow at $I = 2$. Obtained patterns of vortex distribution (Fig. A.20) indicate useful decrease of oscillatory motion of circulations zones in the wake. Installation of the plate causes narrowing of the wake and decreasing of intensity of the vortex beams attaching to the flow from the nearest wake. Respectively hydrodynamic loads decrease [GG05]. Experiments indicate that at increasing of Reynolds number this positive effect decreases though at large numbers Re installation of the separating plate results in noticeable drag decrease (at 10–20%).

### A.7 Conclusions

The numerical algorithm for modeling the laminar flows of viscous incompressible fluid, particularly for non-stationary detached flow past the non-streamline bodies, is developed. The algorithm is based on usage of Navier–Stokes equation system with the “vorticity–velocity” formulation. This algorithm has the following advantages: the possibility of the significant decrease of dimensions of the computation
domain (we carry out the computations within the grid elements with nonzero vorticity values), high accuracy of the boundary conditions fulfillment in the corner points of the body surface and also high-accuracy of vortex value determination in the neighborhood of such points.

Algorithm was approbated on the two-dimensional diffusion problems for a vortex point in a viscous fluid, problems of longitudinal flow past a plate and problems of cross flow past a square prism. Computation result was compared with well-known analytical solutions, numerical simulation results carried out by other authors and also with experimental data. This computations showed that to provide the accuracy of numerical modeling grid spatial steps $\Delta x$, $\Delta y$ and a time step $\Delta t$ must fulfill certain conditions, for example, near the body surface

$$
\Delta x, \Delta y \approx (1 \div 10) \frac{1}{Re}, \quad \frac{\Delta x}{\Delta t} \leq V_{max},
$$

(where $V_{max}$ is the maximal local fluid velocity in the domain). Analysis of obtained results indicate effectiveness of the developed numerical algorithm for modeling the separated flows at moderate Reynolds numbers, particularly, non-stationary processes of vortex structure generation near the body surface and their dynamics in the wake past a non-streamline body.
Performed computations showed that forming of the wake past the body and the body drag essentially depend on the interaction of the separated circulation zones attaching directly to the body surface. Decreasing of non-stationary motion of such zones is accompanied by correspondent decrease of hydrodynamic loads. Basing on this investigation results two control algorithms for a flow past the square prism were considered. In the first case on the front (windward) side of the prism two symmetrical (control) plates were installed. In the second case on the back (leeward) side of the prism one plate was installed (separating vortex sheets with circulations opposite in direction). The analysis of computation results showed that at certain (optimal) parameters of the plates separated circulation zones near the body surface are stable. Non-stationary motion of such zones is decreasing, vorticity generation processes on the body slow down causing decrease of hydrodynamic drag and lateral force acting on the body. Obtained results may be useful when developing new methods of control over flows past a body.

The numerical scheme for solving Navier–Stokes equations represented and validated above has been applied to calculation fluid flows in multiply-connected domains. That allowed deriving regularities referring to generation of vortex wakes near systems of some bodies. Paper [GG08] deals with modelling the laminar flow past two square prisms in tandem. The obtained results show that evolution of the flow and hydrodynamic loads on the bodies depend on the flow pattern in the region between the cylinders. Depending on the spacing between the bodies the flow pattern may be symmetrical, when a stable vortex pare is there formed; non-symmetrical, with separation of large vortices from the front cylinder; and bifurcational, when a sudden jump from the first flow regime to another is possible. Special attention is paid to the effect of external disturbances on flow evolution. Wake behavior behind a parallel pair of square prisms, placed side-by-side normal to a uniform flow, is considered in paper [GG09]. The wake patterns are shown to depend strongly on the gap size between the bodies. When the space is small enough comparatively to the prism side, the wake resembles that seen behind a single body. At gap widths closed to the prism side the flip-flopping vortex shedding pattern where the wake flip-flops randomly between two states is observed. In this regime, the narrow and wide wakes with separate vortex shedding frequencies are formed behind the body system. For a large spacing, two synchronized vortex shedding patterns, antiphase and in-phase, are possible. At very large spaces two Karman vortex streets are generated behind each cylinder. The results obtained are used to ground the scheme of flow control near a square prism that utilizes the small gap in the center. The optimal characteristics of the control scheme are derived. In paper [VVGG09], the kinematics of flow in the system of ten staggered circular cylinders is calculated. The results show that interactions between the wake, which forms behind the entire construction, and the vortices generated near the cylinders in the system cause bifurcation in the vortical flow and appearance of the structures that are convected downstream in the gaps between the cylinders. The calculation results are compared with data of the physical experiments.
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